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THE ABSOLUTE GALOIS GROUP OF A PSEUDO REAL CLOSED FIELD WITH FINITELY MANY ORDERS

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Contents

Introduction

In his celebrated paper [1] on the elementary theory of finite fields Ax considered fields K with the property that every absolutely irreducible variety defined over K has K-rational points. These fields have been later called *pseudo algebraically closed* (pac) by Frey [10] and also *regularly closed* by Ershov [8], and extensively studied by Jarden, Ershov, Fried, Wheeler and others, culminating with the fundamental works [7] and [11].

The above definition of pac fields can be put into the following equivalent version: K is existentially complete (ec), relative to the customary language of fields, into each regular field extension of K. It has been this characterization of pac fields which the author extended in [2] to ordered fields. An ordered field $(K, \leq)$ is called in [2] pseudo real closed (prc) if $(K, \leq)$ is ec in every ordered field extension $(L, \leq)$ with L regular over K. The concept of prc ordered field has also been introduced

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by McKenna in his thesis [14], by analogy with the original algebraic-geometric definition of pac fields.

Recently, Prestel [17] introduced a very inspired concept which extends the concept of a pac field as well as of a prc ordered field. According to [17], a field K is said to be *prc* if K is ec, relative to field language, in every regular field extension of K to which all orders of K extend.

A system $\mathbf{K}=(K;P_1,\dots,P_e)$, where K is a field, e is a positive integer and $P_1,\dots,P_e$ are orders of K (identified with the corresponding positive cones), is called an *e-fold ordered field* (e-field). It turns out by [17, Theorem 1.7] that an e-field $\mathbf{K}$ is ec, relative to the first-order language of e-fields, in every regular e-field extension of $\mathbf{K}$ iff K is prc, $P_i \neq P_j$ for $i \neq j$ and K has exactly e orders. Let us call such an e-field $\mathbf{K}$ a *prc e-field.*

It is well known that the absolute Galois group $G(K)$ of a pac field K is a projective profinite group (see [1] for perfect pac fields). It is also known [16], [7, Proposition 38], that all projective profinite groups occur as $G(K)$, K a pac field. The main goal of the present paper is to prove that the statements above remain true for prc e-fields $\mathbf{K}$ if we replace the absolute Galois group $G(K)$ by a suitable generalization $\mathbf{G}(\mathbf{K})$ called the absolute Galois e-structure of the e-field $\mathbf{K}$, and projectivity for profinite groups by projectivity for the so called profinite e-structures.

Theorem I. *Let* $\mathbf{K}$ *be a prc e-field. Then its absolute Galois e-structure* $\mathbf{G}(\mathbf{K})$ *is a projective profinite e-structure.*

Theorem II. *The necessary and sufficient condition for a profinite e-structure* $\mathbf{G}$ *to be realized as the absolute Galois e-structure over some prc e-field is that* $\mathbf{G}$ *is projective.*

In order to prove the theorems above we introduce and investigate in Sections 1–4 some group-theoretic objects called *e-structures*. Some basic facts concerning the model theory of profinite e-structures are developed in Sections 2,3 on the line of the cologic for profinite groups from [7]. The projective profinite e-structures are characterized in Section 4.

Section 5 answers the question: what is the appropriate extension to the theory of e-fields of the basic concept of Galois group from the field theory? [appropriate in the sense that it must reflect the Galois group structure as well as the relation between this one and the orders of a given e-field]. The answer to this question is suggested by the concept of *order-pair* introduced in [13]. It turns out that the suitable group-theoretic concept for e-fields is the concept of profinite e-structure introduced in Section 1. To each e-field $\mathbf{K}$ we naturally assign a profinite e-structure $G(K)$, called the *absolute Galois e-structure* of $\mathbf{K}$, in such a way that the elementary statements about $\mathbf{G}(\mathbf{K})$ are interpretable in the first-order language of $\mathbf{K}$.

Finally. the proofs of the main results stated above are given in Section 6.

1. Profinite *e*-structures

1.1. Let us fix a natural number e. By an *e-structure* we mean a system $\mathbf{G} = (G; X_1, \dots, X_e)$, where G is a group and the X_i's are non-empty G-sets satisfying the next conditions:

(i) The actions $X_i \times G \to X_i : (x, \tau) \mapsto x^\tau$ are transitive, i.e. the X_i's are G-orbits.

(ii) For $x \in X_i$, $i = 1, \dots, e$, the invariant subgroup $\mathrm{Inv}(x) = \{\tau \in G \mid x^\tau = x\}$ is cyclic of order 2.

If $x \in \bigcup_{i=1}^e X_i$, denote by $\sigma(x)$ the involution of G which generates $\mathrm{Inv}(x)$.

Given an e-structure $\mathbf{G}$, we usually denote by G the underlying group of $\mathbf{G}$ and by $X_i(\mathbf{G})$, $i = 1, \dots, e$, the corresponding G-sets.

A morphism of e-structures from $\mathbf{G}$ to $\mathbf{H}$ is an $(e+1)$-tuple $\varphi = (\varphi^0, \varphi^1, \dots, \varphi^e)$, where $\varphi^0 : G \to H$ is a group morphism and $\varphi^i : X_i(\mathbf{G}) \to X_i(\mathbf{H})$, $i = 1, \dots, e$, are maps subject to the following conditions:

(i) $\varphi^i(x^\tau) = \varphi^i(x)^{\varphi^0(\tau)}$ for $x \in X_i(\mathbf{G})$, $\tau \in G$.

(ii) $\varphi^0(\sigma(x)) = \sigma(\varphi^i(x))$ for $x \in X_i(\mathbf{G})$.

Usually we denote by the same letter, say φ, the maps $\varphi^0, \varphi^1, \dots, \varphi^e$ defining a morphism of e-structures.

Call $\varphi : \mathbf{G} \to \mathbf{H}$ a mono (epi) if $\varphi^0 : G \to H$ is injective (surjective).

A *sub-e-structure* of $\mathbf{G}$ is an e-structure $\mathbf{H}$, where H is a subgroup of G and $X_i(\mathbf{H})$ is a subset of $X_i(\mathbf{G})$, $i = 1, \dots, e$, subject to

(i) There is $x_i \in X_i(\mathbf{G})$ such that $\sigma(x_i) \in H$, $i = 1, \dots, e$.

(ii) $X_i(\mathbf{H}) = \{x_i^\tau \mid \tau \in H\}$ with x_i as above, and the action of H on $X_i(\mathbf{H})$ is induced by the action of G on $X_i(\mathbf{G})$, $i = 1, \dots, e$.

A *quotient e-structure* of $\mathbf{G}$ is an e-structure $\mathbf{E}$ where $E = G/N$ for some normal subgroup N of G with $\sigma(x) \notin N$ for $x \in \bigcup_{i=1}^e X_i(\mathbf{G})$, $X_i(\mathbf{E}) = X_i(\mathbf{G})/N$ is the quotient set w.r.t. the next equivalence relation induced by N:

$$x \sim x' \quad \Leftrightarrow \quad (\exists \tau \in N)\, x' = x^\tau,$$

for $x, x' \in X_i(\mathbf{G})$, $i = 1, \dots, e$, and the actions of E on the $X_i(\mathbf{E})$'s are induced by the actions of G on the $X_i(\mathbf{G})$'s.

If $\varphi : \mathbf{G} \to H$ is a morphism of e-structures, then the image $\varphi(\mathbf{G}) = (\varphi^0(G); \varphi^1(X_1(\mathbf{G})), \dots, \varphi^e(X_e(\mathbf{G})))$ is a sub-e-structure of $\mathbf{H}$ and $\varphi(\mathbf{G}) \cong \mathbf{G}/\mathrm{Ker}\, \varphi^0$.

Let $\varphi : \mathbf{G} \to \mathbf{H}$, $\varphi' : \mathbf{G}' \to \mathbf{H}$ be morphisms of e-structures and assume that the sets $X_i(\mathbf{G}) \times_{X_i(\mathbf{H})} X_i(\mathbf{G}')$, $i = 1, \dots, e$, are non-empty. Then

$$\mathbf{G} \times_{\mathbf{H}} \mathbf{G}' = (G \times_H G'; X_1(\mathbf{G}) \times_{X_1(\mathbf{H})} X_1(\mathbf{G}'), \dots, X_e(\mathbf{G}) \times_{X_e(\mathbf{H})} X_e(\mathbf{G}'))$$

with the canonical morphisms $p : \mathbf{G} \times_{\mathbf{H}} \mathbf{G}' \to \mathbf{G}$, $p' : \mathbf{G} \times_{\mathbf{H}} \mathbf{G}' \to \mathbf{H}'$ is a pullback of the pair (φ, φ').

An e-structure $\mathbf{G}$ is called finite (profinite) if the underlying group G is finite (profinite). By morphisms, monos, epis of profinite e-structures we understand continuous morphisms, monos, epis. By a sub-e-structure $\mathbf{H}$ of a profinite e-structure $\mathbf{G}$ we mean a sub-e-structure of $\mathbf{G}$ for which H is a closed subgroup of G.

The simplest example of e-structure denoted by $\mathbb{Z}_2$ has $\mathbb{Z}/2\mathbb{Z}$ as underlying group which acts trivially on the singletons $X_i(\mathbb{Z}_2)=\{*\}$, $i=1,\dots,e$. $\mathbb{Z}_2$ has no proper sub-e-structures and quotient e-structures.

1.2. Denote by e-**FIN** (e-**PROFIN**) the category of finite (profinite) e-structures. let e-**FINE** (e-**PROFINE**) the subcategory of e-**FIN** (e-**PROFIN**) with the same objects, but only with epis.

Now we extend the duality for profinite groups from [7, §2] to profinite e-structures.

Definition. A (directed) *projective system* (of finite e-structures) is a contravariant functor $\mathfrak{G}$ from a directed non-empty partial ordered set $(\Lambda, \leq)$ to e-**FINE**:

$$\alpha\in\Lambda \mapsto \mathfrak{G}_\alpha,$$

$$\alpha,\beta\in\Lambda,\ \alpha\leq\beta,\ \mapsto \prod_{\alpha,\beta} : \mathfrak{G}_\beta\to\mathfrak{G}_\alpha.$$

Definition. Let $\mathfrak{G}:(\Lambda,\leq)^0\to e$-**FINE** and $\mathfrak{H}:(\Gamma,\leq)^0\to e$-FINE be projective systems. A morphism from $\mathfrak{G}$ to $\mathfrak{H}$ is a pair (φ,ψ), where $\varphi:(\Lambda,\leq)\to(\Gamma,\leq)$ is a monotone map and $\psi:\mathfrak{H}\to\mathfrak{G}$ is a natural transformation such that for each $\alpha\in\Lambda$, the morphism $\Psi_\alpha:\mathfrak{H}_{\varphi(\alpha)}\to\mathfrak{G}_\alpha$ is mono.

Definition. The projective system $\mathfrak{G}:(\Lambda,\leq)^0\to e$-FINE is *complete* if for every $\alpha\in\Lambda$ and every normal subgroup N of $\mathfrak{G}_\alpha$ with $\sigma(x)\notin N$ for $x\in\bigcup_{i=1}^e X_i(\mathfrak{G}_\alpha)$, there exists a unique $\beta\in\Lambda$ such that $\beta\leq\alpha$ and $N=\operatorname{Ker}\prod^0_{\beta,\alpha}$ (it follows that $\mathfrak{G}_\alpha/N$ is a quotient e-structure of $\mathfrak{G}_\alpha$ and $\mathfrak{G}_\beta\cong\mathfrak{G}_\alpha/N$).

Denote by e-**CPS** the category of complete projective systems (of finite e-structures) with morphisms defined as above. Let e-**CPSI** be the subcategory of e-**CPS** with the same objects, but only with morphisms $(\varphi,\psi):\mathfrak{G}\to\mathfrak{H}$ such that φ is injective and ψ is a natural isomorphism.

1.2.1. Proposition. *There exists a canonical duality between the categories e-***PROFIN** *and e-***CPS,** *which induces a duality between e-***PROFINE** *and e-***CPSI.**

Proof. Define a functor $\mathbf{S}:e$-**PROFIN**$\to$(e-**CPS**)^0 as follows. If $\mathbf{G}$ is a profinite e-structure, denote by $\Lambda=\Lambda(\mathbf{G})$ the set of open normal subgroups N of G with $\sigma(x)\notin N$ for $x\in\bigcup_{i=1}^e X_i(\mathbf{G})$. Consider the partial order on Λ defined by $N\leq N'$ iff $N'\subset N$. Λ is cofinal in the set of all open subgroups of G.

Let $\mathbf{S}(\mathbf{G}):(\Lambda,\leq)^0\to e$-**FINE** be the functor given by

$$N\in\Lambda\mapsto \mathbf{G}/N,$$

$$N\leq N'\mapsto \mathbf{G}/N' \xrightarrow{\pi_{N,N'}} \mathbf{G}/N \text{ the canonical epi.}$$

Obviously, $\boldsymbol{S}(\mathbf{G})$ is a complete projective system of finite e-structures.

Given a morphism $\lambda:\mathbf{G}\to\mathbf{H}$ in e-**PROFIN**, let $\boldsymbol{S}(\lambda)=(\varphi,\psi):\boldsymbol{S}(\mathbf{H})\to\boldsymbol{S}(\mathbf{G})$ be the morphism in e-**CPS** defined by

$$\varphi:\Lambda(\mathbf{H})\to\Lambda(\mathbf{G}):\quad N\mapsto\lambda^{-1}(N);$$

$$\psi_N:\mathbf{G}/\lambda^{-1}(N)\to\mathbf{H}/N,\ \text{the canonical mono induced by } \lambda,$$

for $N\in\Lambda(\mathbf{H})$.

Conversely, define a functor $\boldsymbol{G}:(e\text{-}\mathbf{CPS})^0\to e\text{-}\mathbf{PROFIN}$, as follows. If $\mathfrak{G}:(\Lambda,\le)^0\to e$-**FINE** is an object of e-**CPS**, let $\boldsymbol{G}(\mathfrak{G})$ be the profinite e-structure $\varprojlim_{\alpha\in\Lambda}\mathfrak{G}_\alpha$. Given a morphism (φ,ψ) in e-**CPS** from $\mathfrak{G}:(\Lambda,\le)^0\to e$-**FINE** to $\mathfrak{H}:(\Gamma,\le)^0\to e$-**FINE** we get a canonical morphism $\boldsymbol{G}(\varphi,\psi):\boldsymbol{G}(\mathfrak{H})\to\boldsymbol{G}(\mathfrak{G})$ of profinite e-structures, associated to (φ,ψ).

It is a simple exercise to verify that the pair $(\boldsymbol{S},\boldsymbol{G})$ defines a duality between e-**PROFIN** and e-**CPS** which induces a duality between e-**PROFINE** and e-**CPSI**, as contended. □

2. The cologic for profinite e-structures

We develop in this section a cologic for profinite e-structures on the line of the cologic for profinite groups [7, §2].

First we define auxiliary first-order structures dual to profinite e-structures.

Definition. A *projective system of* (discrete) *e-structures* is a contravariant functor $\mathfrak{G}$ defined on a directed partial ordered set $(\Lambda,\le)$ with values in the category of (discrete) e-structures with epis:

$$\alpha\in\Lambda\mapsto\mathfrak{G}_\alpha,$$

$$\alpha\le\beta\mapsto\prod_{\alpha,\beta}:\mathfrak{G}_\beta\to\mathfrak{G}_\alpha.$$

In terms of predicate calculus, a projective system of e-structures is a set S together with the following data:

(i) A subset Λ of S and a directed partial order $\le$ on Λ.

(ii) Some subsets $G, X_1,\dots,X_e$ of S such that S is the disjoint union $\Lambda\,\dot\cup\, G\,\dot\cup\,\bigcup_{i=1}^e X_i$.

(iii) A binary relations on S which defines a map $s:G\cup\bigcup_{i=1}^e X_i\to\Lambda$ in such a way that the restriction maps $s_0:G\to\Lambda$, $s_i:X_i\to\Lambda$, $i=1,\dots,e$ are onto; denote $G_\alpha=s_0^{-1}(\alpha)$, $X_{i,\alpha}=s_i^{-1}(\alpha)$, $i=1,\dots,e$, $\alpha\in\Lambda$.

(iv) A ternary relation on S which defines for each $\alpha\in\Lambda$ a group law $\cdot$ on G_α.

(v) A ternary relation on S which defines for each $\alpha\in\Lambda$ some maps $X_{i,\alpha}\times G_\alpha\to X_{i,\alpha}$, $i=1,\dots,e$ in such a way that $\mathfrak{G}_\alpha=(G_\alpha;X_{1,\alpha},\dots,X_{e,\alpha})$ becomes an e-structure.

(vi) A binary relation on S which defines for arbitrary $\alpha, \beta \in \Lambda$, $\alpha \leq \beta$, an epi of e-structures $\Pi_{\alpha,\beta}: \mathfrak{G}_\beta \to \mathfrak{G}_\alpha$, in such a way that the maps $\alpha \mapsto \mathfrak{G}_\alpha$ and $\alpha \leq \beta \mapsto \Pi_{\alpha,\beta}$ define a contravariant functor $\mathfrak{G}$ on $(\Lambda, \leq)$ with values in the category of e-structures with epis.

Let $\boldsymbol{L}_e$ be the first-order language for such structures. Clearly the class of projective systems of e-structures is axiomatizable in $\boldsymbol{L}_e$ by finitely many $\forall\exists$-sentences. Note that an $\boldsymbol{L}_e$-embedding doesn't define always a morphism of projective systems.

Adjoin to $\boldsymbol{L}_e$ unary predicates R_n for all positive integers n to get a language L_e'.

Definition. A *stratified* projective system of e-structures is an $\boldsymbol{L}_e'$-structure $(S; R_n, n \geq 1)$ where S is a projective system of e-structures (seen as an $\boldsymbol{L}_e$-structure) and for each positive integer n,

$$R_n = \Lambda^{(n)} \cup \bigcup_{\alpha \in \Lambda^{(n)}} \left(G_\alpha \cup \bigcup_{i=1}^{e} X_{i,\alpha} \right), \quad \text{with } \Lambda^{(n)} = \{\alpha \in \Lambda \mid (G_\alpha : 1) \leq n\}.$$

The *rank* of an element $a \in S$ is the smallest $n \in \mathbb{N}$, if such n exists, subject to $a \in R_n$. Otherwise we say that a has infinite rank.

Definition. The *ranked part* $S^{(\omega)}$ of S is the $\boldsymbol{L}_e'$-substructure of S containing only the elements of S with finite rank.

If $S^{(\omega)}$ is non-empty, then $S^{(\omega)}$ represents the maximal projective system (not necessarily directed) of finite e-structures contained in S.

Definition. A stratified projective system S is *ranked* if $S = S^{(\omega)}$, i.e. the $\boldsymbol{L}_e'$-structure S represents a directed projective system of finite e-structures.

Definition. A stratified projective system S is *complete* if the projective system of finite e-structures represented by $S^{(\omega)}$ is directed and complete (see (1.2)), i.e. the next conditions are satisfied:

(i) For $n \geq 1$, $\alpha \in \Lambda^{(n)}$ and N a normal subgroup of G_α with $\sigma(x) \notin N$ for $x \in \bigcup_{i=1}^{e} X_{i,\alpha}$, there exists uniquely $\beta \in \Lambda$ such that $\beta \leq \alpha$ and $N = \operatorname{Ker} \Pi_{\beta,\alpha}$.

(ii) For $n \geq 1$, $\alpha, \beta \in \Lambda^{(n)}$ there is $\gamma \in \Lambda^{(n^2)}$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.

The class of complete projective systems is $\boldsymbol{L}_e'$-axiomatizable.

It follows that the category of complete ranked projective systems with $\boldsymbol{L}_e'$-embeddings may be identified with the category e-**CPSI** introduced in (1.2), the dual of e-**PROFINE**, by (1.2.1). We now use the duality (1.2.1) to extend the cologic for profinite groups developed in [7, §2] to a cologic for profinite e-structures.

We work with a fragment of the logic $\boldsymbol{L}_e'$. The set of bounded $\boldsymbol{L}_e'$-formulas is defined as the smallest set of $\boldsymbol{L}_e'$-formulas containing the atomic formulas, closed under logical connectives, and closed under

$$\Phi \mapsto (\exists x)(R_n(x) \wedge \phi)$$

where $n \in \mathbb{N}$ and x is a variable.

The next lemma is immediate.

2.1. Lemma. *Let S be a stratified projective system, $\phi(x_1, \dots, x_m)$ a bounded L'_e-formula, and $a_1, \dots, a_m \in S^{(\omega)}$. Then*

$$S \models \Phi(a_1, \dots, a_m) \quad \textit{iff} \quad S^{(\omega)} \models \phi(a_1, \dots, a_m).$$

Definitions. (a) A *coformula* (*consentence*) for profinite e-structures is a bounded L'_e-formula (L'_e-sentence).

(b) For an L'_e-structure S, the language $L'_e(S)$ is the augmentation of L'_e by constants for S. We get the obvious notion of *bounded $L'_e(S)$-formula.*

(c) A coformula over a profinite e-structure $\mathbf{G}$ is a bounded $L'_e(\boldsymbol{S}(\mathbf{G}))$-formula (see (1.2) for definition of the functor $\boldsymbol{S}$).

(d) Let $\phi(x_1, \dots, x_m)$ be a coformula over $\mathbf{G}$ and let $a_1, \dots, a_m \in \boldsymbol{S}(\mathbf{G})$. $\mathbf{G}$ *cosatisfies* $\phi(a_1, \dots, a_m)$ (written $\mathbf{G} \dashv \phi(a_1, \dots, a_m)$) if $\boldsymbol{S}(\mathbf{G}) \models \phi(a_1, \dots, a_m)$.

(e) The *cotheory* of $\mathbf{G}$ (written Coth($\mathbf{G}$)) is the set of all cosentences cosatisfied by $\mathbf{G}$.

(f) $\mathbf{G}$ and $\mathbf{H}$ are *coequivalent* if Coth($\mathbf{G}$) = Coth($\mathbf{H}$).

(g) An epi $\varphi : \mathbf{G} \to \mathbf{H}$ is *coelementary* if the corresponding L'_e-embedding $\boldsymbol{S}(\varphi) : \boldsymbol{S}(\mathbf{H}) \to \boldsymbol{S}(\mathbf{G})$ is b-elementary, i.e. $\boldsymbol{S}(\varphi)$ preserves bounded $L'_e(\boldsymbol{S}(\mathbf{H}))$-sentences.

3. Co-ultraproducts of profinite *e*-structures

Let $(\mathbf{G}_\lambda)_{\lambda \in \Gamma}$ be a family of profinite e-structures and D be an ultrafilter on Γ. For each $\lambda \in \Gamma$, $\Lambda_\lambda = \Lambda(\mathbf{G}_\lambda)$ is the set of open normal subgroups N of G_λ for which $\sigma(x) \notin N$ for $x \in \bigcup_{i=1}^{e} X_i(\mathbf{G})$. If $N \in \Lambda_\lambda$, then $\mathbf{G}_\lambda/N$ is the finite quotient e-structure of $\mathbf{G}_\lambda$ determined by N. Λ_λ is partially ordered by the relation $N \leq N'$ iff $N' \subset N$.

Form the L'_e-structure $\prod_{\lambda \in \Gamma} \boldsymbol{S}(\mathbf{G}_\lambda)/D$. This ultraproduct is a complete stratified projective system of (discrete) e-structures, but is not necessarily ranked. In a functorial setting, $\prod_{\lambda \in \Gamma} \boldsymbol{S}(\mathbf{G}_\lambda)/D$ is a contravariant functor defined on the directed partially ordered set $\prod_{\lambda \in \Gamma}(\Lambda_\lambda, \leq)/D$ with values in the category of (discrete) e-structures with epis, defined on objects as follows:

$$(N_\lambda)/D \mapsto \prod_{\lambda \in \Gamma} (\mathbf{G}_\lambda/N_\lambda)/D.$$

Denote by $\prod^\omega \boldsymbol{S}(\mathbf{G}_\lambda)/D$ the ranked part $(\prod \boldsymbol{S}(\mathbf{G}_\lambda)/D)^{(\omega)}$ of $\prod \boldsymbol{S}(\mathbf{G}_\lambda)/D$. Then $\prod^\omega \boldsymbol{S}(\mathbf{G}_\lambda)/D$ is a (directed) complete projective system of finite e-structures. The next lemma follows easily from (2.1) and Løs' Theorem.

3.1. Lemma. *For each bounded L'_e-formula $\phi(x_1, \dots, x_m)$ and arbitrary $f_1, \dots, f_m \in \prod \boldsymbol{S}(\mathbf{G}_\lambda)$ with $f_1/D, \dots, f_m/D \in \prod^\omega \boldsymbol{S}(\mathbf{G}_\lambda)/D$, the next statements are equivalent:*

(i) $\prod^\omega \boldsymbol{S}(\mathbf{G}_\lambda)/D \models \phi(f_1/D, \dots, f_m/D)$,

(ii) $\{\lambda \in \Gamma \mid \boldsymbol{S}(\mathbf{G}_\lambda) \models \phi(f_1(\lambda), \dots, f_m(\lambda))\} \in D.$

Define the co-ultraproduct $\prod^\omega \mathbf{G}_\lambda/D$ as the profinite e-structure $\boldsymbol{G}(\prod^\omega \boldsymbol{S}(\mathbf{G}_\lambda)/D)$ corresponding by duality to the complete projective system of finite e-structures $\prod^\omega \boldsymbol{S}(\mathbf{G}_\lambda)/D$. Moreover, we get obviously a covariant functor $\prod^\omega/D: e\text{-}\mathbf{PROFIN}^\Gamma \to e\text{-}\mathbf{PROFIN}$ inducing by restriction a covariant functor $\prod^\omega/D: e\text{-}\mathbf{PROFINE}^\Gamma \to e\text{-}\mathbf{PROFINE}$. For $\mathbf{G}_\lambda = \mathbf{G}$ for all $\lambda \in \Gamma$, we write $\mathbf{G}^{\omega\Gamma}/D$ instead of $\prod^\omega \mathbf{G}_\lambda/D$, and call this profinite e-structure the co-ultrapower of $\mathbf{G}$ w.r.t. the pair (Γ, D). Thus we get a covariant functor ${}^{\omega\Gamma}/D: e\text{-}\mathbf{PROFIN} \to e\text{-}\mathbf{PROFIN}$ inducing by restriction a covariant functor ${}^{\omega\Gamma}/D: e\text{-}\mathbf{PROFINE} \to e\text{-}\mathbf{PROFINE}$. The diagonal map $\Delta: \boldsymbol{S}(\mathbf{G}) \to \boldsymbol{S}(\mathbf{G})^\Gamma/D$ induces by (3.1) a b-elementary map $\Delta: \boldsymbol{S}(\mathbf{G}) \to (\boldsymbol{S}(\mathbf{G})^\Gamma/D)^{(\omega)}$ and by duality a coelementary epi $\nabla: \mathbf{G}^{\omega\Gamma}/D \to \mathbf{G}$.

We end this section with a construction which is useful in Section 6. Let $\mathbf{G}$ be a profinite e-structure and let Γ be a cofinal subset of the directed partially ordered set $\Lambda(\mathbf{G})$ of open normal subgroups N of G with $\sigma(x) \notin N$ for $x \in \bigcup_{i=1}^e X_i(\mathbf{G})$. Obviously $\mathbf{G} \cong \varprojlim_{N \in \Gamma} \mathbf{G}/N$. Consider the family of sets $Z_N = \{U \in \Gamma \mid N \leq U\} = \{U \in \Gamma \mid U \subset N\}$, for all $N \in \Gamma$. Since Γ is cofinal in $\Lambda(\mathbf{G})$, the family $(Z_N)_{N \in \Gamma}$ is a filter basis on Γ. Let D be an ultrafilter on Γ containing the Z_N's for all $N \in \Gamma$. Consider the canonical epis $\pi_N: \mathbf{G} \to \mathbf{G}/N$ for $N \in \Gamma$ and define the $\boldsymbol{L}_e'$-embedding $\lambda: \boldsymbol{S}(\mathbf{G}) \to \prod_{N \in \Gamma} \boldsymbol{S}(\mathbf{G}/N)/D$ induced by the canonical monotone map

$$\lambda': (\Gamma, \leq) \to \prod_{N \in \Gamma} \Lambda(\mathbf{G}/N)/D: \quad U \mapsto (UN/N)/D.$$

Clearly λ' is injective and for each $U \in \Gamma$, the canonical morphism $\mathbf{G}/U \to \prod_{N \in \Gamma} (\mathbf{G}/UN)/D$ is an isomorphism since $\prod_{N \in \Gamma} (\mathbf{G}/UN)/D \cong (\mathbf{G}/U)^\Gamma/D \cong \mathbf{G}/U$ as $\mathbf{G}/U$ is finite.

The $\boldsymbol{L}_e'$-embedding induces by restriction to ranked parts the $\boldsymbol{L}_e'$-embedding

$$\lambda: \boldsymbol{S}(G) \to \prod_{N \in \Gamma}^\omega \boldsymbol{S}(\mathbf{G}/N)/D.$$

By duality we get a canonical epi of profinite e-structures

$$\boldsymbol{G}(\lambda): \prod_{N \in \Gamma}^\omega (\mathbf{G}/N)/D \to \mathbf{G}.$$

4. Projective profinite e-structures

A profinite e-structure $\mathbf{G}$ is *projective* if every diagram of profinite e-structures

$$\begin{array}{ccc} & & \mathbf{G} \\ & & \downarrow{\scriptstyle\varphi} \\ \mathbf{E} & \xrightarrow{\psi} & \mathbf{H} \end{array} \tag{1}$$

with ψ epi, can be completed to a commutative diagram by a morphism $\theta:\mathbf{G}\to\mathbf{E}$. (We say that the extension problem (1) has a solution θ).

In the following we give a characterization of projective profinite e-structures.

4.1. Proposition. *Let* $\mathbf{G}$ *be a profinite e-structure. The next statements are equivalent:*

(i) $\mathbf{G}$ *is projective.*

(ii) *Every epi* $\psi:\mathbf{E}\to\mathbf{G}$ *splits, i.e. there is* $i:\mathbf{G}\to\mathbf{E}$ *with* $\psi i=1_{\mathbf{G}}$.

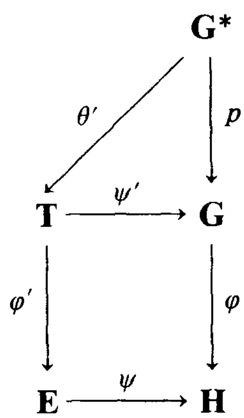
(iii) *For each epi* $\psi:\mathbf{E}\to\mathbf{G}$ *there exist a coelementary epi* $p:\mathbf{G}^*\to\mathbf{G}$ *and a morphism* $\theta:\mathbf{G}^*\to\mathbf{E}$ *such that* $p=\psi\theta$.

(iv) *Every extension problem* (1) *with* φ,ψ *epis and* $\mathbf{E}$ *finite has a solution* $\theta:\mathbf{G}\to\mathbf{E}$.

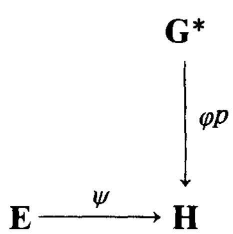
Proof. (i)$\to$(ii) is trivial.

(ii)$\to$(iii) is immediate. Take $\mathbf{G}^*=\mathbf{G}$ and $p=1_{\mathbf{G}}$.

(iii)$\to$(iv) Consider the diagram (1) with φ,ψ epis, $\mathbf{E}$ finite. By assumption we get a commutative diagram



where $(\mathbf{T};\psi',\varphi')$ with ψ',φ' epis is the pullback of the pair (φ,ψ) and p is a coelementary epi. Now, the existence of a solution θ for the extension problem (1) is obviously equivalent to the fact that $\mathbf{G}$ cosatisfies certain cosentence ϕ over $\mathbf{G}$. Since $\varphi'\theta'$ is a solution of the extension problem derived from (1)



it follows $\mathbf{G}^*\Dashv\phi$. As p is a coelementary epi we get finally $\mathbf{G}\Dashv\phi$.

(iv)$\to$(i). First observe that (iv) is equivalent with the next statement.

(iv′) Every extension problem (1) with ψ epi, $\mathbf{E}$ finite has a solution.

Indeed it suffices to apply (iv) to the extension problem

$$\begin{array}{ccc} & & \mathbf{G} \\ & & \downarrow \varphi \\ \varphi(\mathbf{H}) \times_{\mathbf{H}} \mathbf{E} & \xrightarrow{\psi'} & \varphi(\mathbf{H}) \end{array}$$

where the projection ψ' is epi since ψ is so.

Next consider the diagram (1) with ψ epi and assume that the kernel A of the epi $\psi: E \to H$ is finite. As A is a closed normal subgroup of E, there is an open normal subgroup N of E with $N \cap A = 1$. We may assume $\psi(N) \in \Lambda(\mathbf{H})$, i.e. $\sigma(x) \notin N$ for all $x \in \bigcup_{i=1}^{e} X_i(\mathbf{H})$. We get the canonical commutative diagram

$$\begin{array}{ccc} & & \mathbf{G} \\ & & \downarrow \varphi \\ \mathbf{H} \times_{\mathbf{H}/\psi(N)} \mathbf{E}/N \cong \mathbf{E} & \xrightarrow{\psi} & \mathbf{H} \\ \downarrow \pi' & & \downarrow \pi \\ \mathbf{E}/N & \xrightarrow{\psi'} & \mathbf{H}/\psi(N) \end{array}$$

Since $\mathbf{E}/N$ is finite, we get by (iv′) some $\theta': \mathbf{G} \to \mathbf{E}/N$ with $\pi\varphi = \psi'\theta'$. By universality of pullbacks, there is uniquely $\theta: \mathbf{G} \to \mathbf{E}$ with $\varphi = \psi\theta$ and $\theta' = \pi'\theta$.

Finally, consider an arbitrary diagram (1) with ψ epi, and let S be the set of pairs (N, λ), where N is a closed subgroup of $A = \operatorname{Ker} \psi$ which is normal in E and $\lambda: \mathbf{G} \to \mathbf{E}/N$ is a morphism such that $\varphi = \psi_N \lambda$, with $\psi_N: \mathbf{E}/N \to \mathbf{H}$ induced by ψ. The set S is non-empty since $(A, \varphi) \in S$. Define a partial order on S by: $(N_1, \lambda_1) \le (N_2, \lambda_2)$ iff $N_2 \subset N_1$ and $\lambda_1 = \pi_{N_2, N_1} \lambda_2$, where $\pi_{N_2, N_1}: \mathbf{E}/N_2 \to \mathbf{E}/N_1$ is canonic. S is inductive w.r.t. the order $\le$. Let (N, λ) be a maximal pair in S. If $N \ne 1$, then there exists by [18, Ch.I, Lemma 5], a proper open subgroup N' of N which is normal in E. Then N/N' is finite and so there is $\lambda': \mathbf{G} \to \mathbf{E}/N'$ with $\lambda = \psi'\lambda'$, where $\psi': \mathbf{E}/N' \to \mathbf{E}/N$ is canonic. We get $(N', \lambda') \in S$ and $(N', \lambda') > (N, \lambda)$ contrary to maximality of (N, λ). Consequently $N = 1$ and $\varphi = \psi\lambda$. □

Remark. It is shown in [5, Theorem 3.1] that the statements (i)–(iv) above are also equivalent with the following one.

(v) Every extension problem (1), with $\mathbf{E}$ finite, ψ Frattini cover of $\mathbf{H}$ (i.e. there is no proper sub-e-structure $\mathbf{E}'$ of $\mathbf{E}$ such that the restriction $\psi/\mathbf{E}': \mathbf{E}' \to \mathbf{H}$ is epi) and $A = \operatorname{Ker} \psi$ abelian minimal normal subgroup of E, has a solution.

It is obtained in this way a suitable generalization of a well known characterization of projective profinite groups [12, Proposition 1].

We end this section with a lemma which is useful in Section 6.

4.2. Lemma. *Let* $\mathbf{G}$ *be a projective profinite e-structure. Then* $\mathbb{Z}_2$ *is a quotient e-structure of* $\mathbf{G}$.

Proof. For all $i=1,\dots,e$, fix some $x_i \in X_i(\mathbf{G})$, and let $\sigma_i=\sigma(x_i)$. Let $\mathbf{E}$ be the profinite e-structure with underlying profinite group $E=G\times\mathbb{Z}/2\mathbb{Z}$, and E-sets $X_i(\mathbf{E})=H_i\backslash E$ where H_i is the cyclic group of order 2 of E generated by the involution $(\sigma_i, 1+2\mathbb{Z})$, $i=1,\dots,e$. The action of E on $X_i(\mathbf{E})$ is given by: $(H_i(g,\tau),(g',\tau'))\mapsto H_i(gg',\tau+\tau')$, for $g,g'\in G$, $\tau,\tau'\in\mathbb{Z}/2\mathbb{Z}$. The profinite e-structure $\mathbf{E}$ with the epis $p_1:\mathbf{E}\to\mathbf{G}$, $p_2:\mathbf{E}\to\mathbb{Z}_2$ given by $p_1^0(g,\tau)=g$, $p_2^0(g,\tau)=\tau$, $p_1^i(H_i(g,\tau))=x_i^g$, $p_2^i(H_i(g,\tau))=*$, $i=1,\dots,e$, is a direct product of $\mathbf{G}$ and $\mathbb{Z}_2$. As $\mathbf{G}$ is projective there is a mono $\eta:\mathbf{G}\to\mathbf{E}$ splitting p_1, i.e. $p_1\eta=1_{\mathbf{G}}$. Thus we get a morphism $p_2\eta:\mathbf{G}\to\mathbb{Z}_2$. Since the morphisms of e-structures taking values in $\mathbb{Z}_2$ are epis, we conclude that $\mathbb{Z}_2$ is a quotient e-structure of $\mathbf{G}$. □

5. From *e*-fold ordered fields to profinite *e*-structures

Let $\mathbf{K}=(K;P_1,\dots,P_e)$ be an e-field, $e\geq 1$, and L be a Galois extension of K such that L is not formally real (fr) over the ordered fields (K,P_i), $i=1,\dots,e$. We naturally assign to the pair $(\mathbf{K},L)$ a profinite e-structure $\mathbf{G}(L/\mathbf{K})=(G(L/K);X_1(L/\mathbf{K}),\dots,X_e(L/\mathbf{K}))$ called the *Galois e-structure* of $L/\mathbf{K}$. The underlying group of $\mathbf{G}(L/\mathbf{K})$ is the Galois group $G(L/K)$ of L over K, $X_i(L/\mathbf{K})$ is the set of pairs (σ,Q), σ an involution of $G(L/K)$, Q an order extending P_i on the fixed field $L(\sigma)$, and the action $X_i(L/\mathbf{K})\times G(L/K)\to X_i(L/\mathbf{K})$ is given by $((\sigma,Q),\tau)\mapsto(\sigma^\tau,Q^\tau)$ with $\sigma^\tau=\tau^{-1}\sigma\tau$, $Q^\tau=\{a^\tau:=\tau(a)\mid a\in Q\}$. It follows easily that the invariant subgroup of some $(\sigma,Q)\in X_i(L/\mathbf{K})$ is the cyclic group of $G(L/K)$ generated by the involution σ. Note that $\mathbf{G}(L/\mathbf{K})$ is the projective limit $\varprojlim \mathbf{G}(E/\mathbf{K})$ of finite e-structures, where E ranges over all finite Galois extensions of K with $E\subset L$ and E is not fr over (K,P_i), $i=1,\dots,e$.

In particular, if $L=\tilde{K}$ is the algebraic closure of K, we get the *absolute Galois e-structure* $\mathbf{G}(\mathbf{K})=\mathbf{G}(\tilde{K}/\mathbf{K})$ of the e-field $\mathbf{K}$. Note that $X_i(\mathbf{K})=X_i(\tilde{K}/\mathbf{K})$ is identified with the set of involutions σ of $G(K)=G(\tilde{K}/K)$ for which the fixed field $\tilde{K}(\sigma)$ is a ral closure of (K,P_i), $i=1,\dots,e$.

Denote by $\boldsymbol{F}_e$ the first-order language of e-fields. $\boldsymbol{F}_e$ is an extension of the language $(+,-,\cdot,0,1)$ of rings with e unary predicates $\pi_1,\dots,\pi_e$ standing for orders.

A basic fact is that the cotheory of $\mathbf{G}(\mathbf{K})$, $\mathbf{K}$ an e-field, is interpretable in $\mathbf{K}$, as follows from the next analogue of [7] Lemma 17.

5.1. Proposition. *There is a recursive map* $\phi\mapsto\hat{\phi}$ *from cosentences to* $\boldsymbol{F}_e$*-sentences such that for every cosentence* ϕ *and every e-field* $\mathbf{K}$, $\mathbf{G}(\mathbf{K})\models\phi$ *iff* $\mathbf{K}\models\hat{\phi}$.

Proof. The statement is a consequence of the following facts:

(1) Under the Galois duality $L \mapsto G(L)$, the following objects are in 1-1 correspondence: finite Galois extension L/K, with $[L:K]=m$ and L not fr over (K,P_i), $i=1,\dots,e$, and open normal subgroups $N\in \Lambda(\mathbf{G}(K))$ (i.e. $N\cap\bigcup_{i=1}^{e} X_i(\mathbf{K})$ is empty) with $(G(K):N)=m$.

(2) Coding finite extensions of K in K: For each m, let us fix the basis $(b_1,\dots,b_m)$ of K^m by $b_i=(0,\dots,1,0,\dots,0)$ with 1 on the ith place. Then a point $(c_{ijk})_{i,j,k\le m}=c\in K^{m^3}$ uniquely determines an m-dimensional K-algebra Ac. It follows via the splitting field criterion that the c such that Ac is a Galois extension of K form a first-order definable subset of K^{m^3}. Moreover, the $(c,d)\in K^{m^3}\times K^{n^3}$ for which Ac, Ad are Galois extensions of K and Ac is K-embeddable in Ad form a first-order definable subset of $K^{m^3}\times K^{n^3}$.

(3) For each finite e-structure $\mathbf{G}$, the $c\in K^{m^3}$ for which Ac is a Galois extension of K, not fr over (K,P_i), $i=1,\dots,e$, and $\mathbf{G}(Ac/\mathbf{K})\cong\mathbf{G}$ form an $\boldsymbol{F}_e$-definable subset of K^{m^3}. Indeed, the condition "Ac is not fr over (K,P_i)" is equivalent to the existence of some $z\in Ac$ such that the minimal polynomial of z over K has no roots in the real closure $\overline{(K,P_i)}$ of (K,P_i). On the other hand the condition "the subfield Ad of Ac as above is maximal with the property that Ad is fr over (K,P_i) and there are k distinct orders extending P_i on Ad" is equivalent to the fact that $[Ac:Ad]=2$, $Ad=K[z]$ and the minimal polynomial of z over K has k distinct roots in $\overline{(K,P_i)}$. Note that the statements above may be translated in the language of $(K;P_1,\dots,P_e)$ thanks to elimination of quantifiers for real closed fields. □

The next result is a generalization of [7, Lemma 19].

5.2. Lemma. *Let D be an ultrafilter on the index set Γ, and $\mathbf{K}_\gamma=(K_\gamma;P_{1,\gamma},\dots,P_{e,\gamma})$, $\gamma\in\Gamma$, be e-fields. For each $\gamma\in\Gamma$, let L_γ be a Galois extension of K_γ such that L_γ is not fr over $(K_\gamma,P_{i,\gamma})$, $i=1,\dots,e$. Assume that there exists $m\in\mathbb{N}$ such that for almost all (relative to D) $\gamma\in\Gamma$, there exists a finite Galois extension M_γ of K_γ, contained in L_γ, which is not fr over $(K_\gamma,P_{i,\gamma})$, $i=1,\dots,e$, with $[M_\gamma:K_\gamma]\le m$.*

Denote by $\mathbf{K}=(K;P_1,\dots,P_e)$ the ultraproduct $\prod\mathbf{K}_\gamma/D$ and by L the algebraic closure of K in $\prod L_\gamma/D$. Then L is Galois over K and not fr over (K,P_i), $i=1,\dots,e$, and $\mathbf{G}(L/\mathbf{K})$ is canonically isomorphic to the co-ultraproduct $\prod^a G(L_\gamma/\mathbf{K}_\gamma)/D$.

Proof. The statement follows from the next facts, which are consequences of Løs' theorem and elimination of quantifiers for real closed fields:

(1) A Galois extension of $\prod K_\gamma/D$ of degree n, contained in $\prod L_\gamma/D$, can be identified with some $\prod N_\gamma/D$, where N_γ is a Galois extension of K_γ, contained in L_γ, which is for almost all (relative to D) $\gamma\in\Gamma$ of degree n over K_γ.

(2) In the above, $\prod N_\gamma/D$ is not fr over (K,P_i), $i=1,\dots,e$, iff N_γ is not fr over $(K_\gamma,P_{i,\gamma})$, $i=1,\dots,e$, for almost all $\gamma\in\Gamma$. In this case, the finite e-structure $\mathbf{G}(\prod N_\gamma/D\,|\,\mathbf{K})$ is naturally isomorphic to $\prod\mathbf{G}(N_\gamma/\mathbf{K}_\gamma)/D$. □

5.3. Corollary. *Let D be an ultrafilter on the index set Γ and $\mathbf{K}_\gamma$, $\gamma \in \Gamma$, be e-fields. Then $\mathbf{G}(\prod \mathbf{K}_\gamma / D)$ is canonically isomorphic to $\prod^\omega \mathbf{G}(\mathbf{K}_\gamma)/D$.*

6. Proof of the main results

In order to prove the two main results of the paper we need the following lemma, a non-trivial generalization of [11, Lemma 1.1], [3, II, Lemma 4.1].

6.1. Lemma. *Let $\mathbf{K} = (K; P_1, \dots, P_e)$ be an e-field, L a Galois extension of K which is not fr over (K, P_i), $i = 1, \dots, e$, $\mathbf{G}$ a profinite e-structure and $\psi : \mathbf{G} \to \mathbf{G}(L/\mathbf{K})$ an epi. Then there exist an extension $\mathbf{E} = (E; Q_1, \dots, Q_e)$ of $\mathbf{K}$, with E regular over K, a Galois extension F of E such that L is the algebraic closure of K in F (in particular, F is not fr over (E, Q_i), $i = 1, \dots, e$) and an isomorphism $\eta : \mathbf{G} \to \mathbf{G}(F/\mathbf{E})$ such that the next diagram is commutative*

$$\begin{array}{ccc} \mathbf{G} & \xrightarrow{\ \eta\ } & \mathbf{G}(F/\mathbf{E}) \\ & {\scriptstyle\psi}\searrow \quad \swarrow{\scriptstyle \mathrm{res}} & \\ & \mathbf{G}(L/\mathbf{K}) & \end{array} \tag{1}$$

Proof. (a) First, let us consider the finite case, i.e. assume $\mathbf{G}(L/\mathbf{K})$ and $\mathbf{G}$ are finite. Let $U = \{u^g \mid g \in G\}$ be a set of $|G|$ algebraically independent elements over K. The group G acts on U from the right in an obvious manner. It also acts on L through ψ by the formula $a^g = a^{\psi(g)}$. Consequently, G acts on the field of rational functions $F = L(U)$. Let E be the fixed field of G in F. It follows that $E \cap L = K$ and LE is regular over L, as a subfield of a rational function field over L, and hence E is regular over K. Now let us identify the group G with $G(F/E)$ in the obvious manner and the group epi $\psi : G \to G(L/K)$ with the restriction $\mathrm{res} : G(F/E) \to G(L/K)$. It remains to show that there are some orders Q_i of E such that Q_i extends P_i, $i = 1, \dots, e$, and the identity group isomorphism 1_G extends to an isomorphism $\eta : \mathbf{G} \to G(F/\mathbf{E})$ of e-structures in such a way that the diagram (1) commutes.

Fix some $x_i \in X_i(\mathbf{G})$, $i = 1, \dots, e$, and let $\sigma_i = \sigma(x_i) \in G = G(F/E)$. Then $\psi(x_i) = (\tau_i, P_i')$, where τ_i is an involution of $G(L/K)$ which coincides with the restriction of σ_i on L and P_i' is an order extending P_i on the fixed field $L(\tau_i)$ of τ_i in L. So it suffices to extend P_i' to an order Q_i' on the fixed field $F(\sigma_i)$ of σ_i in F, take the restriction Q_i of Q_i' on E and define

$$\eta(x_i^\lambda) = (\sigma_i^\lambda, Q_i'^\lambda) \quad \text{for } \lambda \in G = G(F/E), \quad i = 1, \dots, e.$$

Fix some $i \in \{1, \dots, e\}$ and let $M = L(\tau_i)$. Then there exists $a \in L \setminus M$ such that $L = M[a]$ and $-a^2 \in P_i'$. σ_i acts obviously on the field of rational functions $M(U)$. Let $N \supset M$ be the fixed field of σ_i in $M(U)$.

First let us show that $F(\sigma_i)=N[a(u^1-u^{\sigma_i})]$. Each element of F can be uniquely written in the form $f+af'$ with $f,f'\in M(U)$. Let $f+af'\in F(\sigma_i)$. Then $f+af'=(f+af')^{\sigma_i}=f^{\sigma_i}-af'^{\sigma_i}$, and hence $f^{\sigma_i}=f$ and $f'^{\sigma_i}=-f'$. Thus we get

$$f+af'=f+a(u^1-u^{\sigma_i})(f'/(u^1-u^{\sigma_i}))\in N[a(u^1-u^{\sigma_i})]$$

since $f\in N$ and $f'/(u^1-u^{\sigma_i})\in N$.

In order to extend P_i to an order Q_i' of $F(\sigma_i)$, it suffices to extend P_i' to an order Q_i'' of N in such a way that $(u^1-u^{\sigma_i})^2\in -Q_i''$. For, if so, then $(a(u^1-u^{\sigma_i}))^2\in Q_i''$, i.e. $F(\sigma_i)$ is fr over (N,Q_i'').

Consider the tower of fields

$$M\subset S\subset N\subset M(U)$$

where

$$S=M(u^\lambda+u^{\lambda\sigma_i},u^\lambda u^{\lambda\sigma_i}\mid\lambda\in G)=M(u^\lambda+u^{\lambda\sigma_i},(u^\lambda-u^{\lambda\sigma_i})^2\mid\lambda\in G).$$

As the transcendency degree of $M(U)/M$ is $|G|$ and $M(U)=S[u^\lambda\mid\lambda\in G]$ with the u^λ algebraic over S, it follows that S/M is purely transcendental and the set $\{u^\lambda+u^{\lambda\sigma_i},(u^\lambda-u^{\lambda\sigma_i})^2\mid\lambda\in G\}$ is a transcendency basis of $M(U)/M$. Consequently, there exists some order Q''' of S such that Q''' extends P_i' and $(u^\lambda-u^{\lambda\sigma_i})^2\in -Q'''$ for all $\lambda\in G$. Let Q''' be such an order. It remains to show that N is fr over (S,Q''').

Let $N'=S[(u^\lambda-u^{\lambda\sigma_i})(u^1-u^{\sigma_i})\mid\lambda\in G]$. Let us show that $N'=N$. The inclusion $N'\subset N$ is trivial, so it remains to verify that $[M(U):N']=2$. Since $[N'[u^1]:N']=2$, it suffices to show that $M(U)=N'(u^1]$. However the latter equality is a consequence of the identities $u^\lambda=\alpha_\lambda+\beta_\lambda u^1$, $\lambda\in G$, with

$$\beta_\lambda=\frac{u^\lambda-u^{\lambda\sigma_i}}{u^1-u^{\sigma_i}}=\frac{(u^\lambda-u^{\lambda\sigma_i})(u^1-u^{\sigma_i})}{(u^1-u^{\sigma_i})^2}\in N',$$

$$\alpha_\lambda=\frac{u^1u^{\lambda\sigma_i}-u^{\sigma_i}u^\lambda}{u^1-u^{\sigma_i}}=\frac{(u^\lambda+u^{\lambda\sigma_i})-\beta_\lambda(u^1+u^{\sigma_i})}{2}\in N'.$$

Thus we get $N'=N$.

Let $\zeta_\lambda=(u^\lambda-u^{\lambda\sigma_i})(u^1-u^{\sigma_i})$, $\lambda\in G$, and so $N=S[\zeta_\lambda\mid\lambda\in G]$. Let us show that the degree of $S[\zeta_\lambda]=S[\zeta_{\lambda\sigma_i}]$ over S is 2 for $\lambda\neq 1$, $\lambda\neq\sigma_i$. Obviously, $\zeta_\lambda^2\in S$. On the other hand, $\zeta_\lambda\notin S$ for $\lambda\neq 1$, $\lambda\neq\sigma_i$, since $u^\lambda-u^{\lambda\sigma_i}$ and $u^1-u^{\sigma_i}$ are algebraically independent over M and the polynomial $W^2-YZ\in M(Y,Z)[W]$ is irreducible. As $\zeta_\lambda^2=[-(u^\lambda-u^{\lambda\sigma_i})^2][-(u^1-u^{\sigma_i})^2]\in Q'''$, the order Q''' of S can be extended to an order of N, as contended.

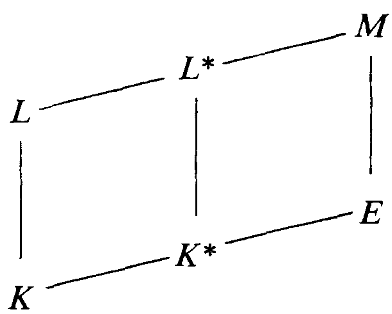
(b) Now let us consider the general case. Let Γ be the subset of $\Lambda(\mathbf{G})$ consisting of those N with $\psi(N)\in\Lambda(\mathbf{G}(L/\mathbf{K})$, i.e. the fixed field L_N of $\psi(N)$ in L is a finite Galois extension of K which is not fre over (K,P_i), $i=1,\dots,e$. Γ is cofinal in $\Lambda(\mathbf{G})$,

$$\mathbf{G}\cong\varprojlim_{N\in\Gamma}\mathbf{G}/N,\qquad \mathbf{G}(L/\mathbf{K})\cong\varprojlim_{N\in\Gamma}\mathbf{G}(L_N/\mathbf{K})\quad\text{and}\quad \psi=\varprojlim_{N\in\Gamma}\psi_N,$$

where the epis $\psi_N:\mathbf{G}/N\to\mathbf{G}(L_N/\mathbf{K})$ are induced by ψ. Using the construction from Section 3 we get a commutative diagram of epis for a suitable ultrafilter D on Γ

$$\begin{array}{ccc} \prod^{\omega}(\mathbf{G}/N)/D & \xrightarrow{\prod^{\omega}\psi_N/D} & \prod^{\omega}\mathbf{G}(L_N/\mathbf{K})/D \\ \Big\downarrow \theta & & \Big\downarrow \theta' \\ \mathbf{G} & \xrightarrow{\quad\psi\quad} & \mathbf{G}(L/\mathbf{K}) \end{array} \tag{2}$$

On the other hand, by the first part of the proof, we get for each $N\in\Gamma$ an extension $\mathbf{E}_N=(E_N;\,Q_{1,N},\dots,Q_{e,N})$ of $\mathbf{K}$ with E_N regular over K, and a finite Galois extension F_N of E_N in such a way that L_N is the algebraic closure of K in E_N, the e-structure $\mathbf{G}(F_N/\mathbf{E}_N)$ is identified with $\mathbf{G}/N$ and the epi ψ_N is identified with the restriction $\mathrm{res}:\mathbf{G}(F_N/\mathbf{E}_N)\to\mathbf{G}(L_N\,|\,\mathbf{K})$. Let $\mathbf{K}^*=(K^*;\,P_1^*,\dots,P_e^*)=\mathbf{K}^{\Gamma}/D$, $\mathbf{E}=(E;\,Q_1,\dots,Q_e)=\prod\mathbf{E}_N/D$, L^* be the algebraic closure of K^* in $\prod L_N/D$ and M be the algebraic closure of E in $\prod E_N/D$. Consider the diagram of fields



(3)

We get easily that the extensions E/K, E/K^*, M/L and M/L^* are regular. Fix some U in Γ and let $(G:U)=m$. Since, by choice of D, $\{V\in\Gamma\,|\,U\le V\}\in D$ it follows that for almost all $N\in\Gamma$, F_N contains a subfield which is Galois over E_N, not fr over $(E_N,Q_{i,N})$, $i=1,\dots,e$, and of degree over E_N bounded by m. Consequently, by (5.2), the Galois extension M of E is not fr over (E,Q_i), $i=1,\dots,e$, and $\mathbf{G}(M/\mathbf{E})$ is canonically isomorphic to $\prod^{\omega}\mathbf{G}(F_N/\mathbf{E}_N)/D\cong\prod^{\omega}(\mathbf{G}/N)/D$. Similarly, the Galois extension L^* of K^* is not fr over (K^*,P_i^*), $i=1,\dots,e$ and $\mathbf{G}(L^*/\mathbf{K}^*)$ is canonically isomorphic to $\prod^{\omega}\mathbf{G}(L_N/\mathbf{K})/D$. From (2) and (3) we get the commutative diagram of epis

$$\begin{array}{ccc} \mathbf{G}(M/\mathbf{E}) & \xrightarrow{\mathrm{res}=\prod^{\omega}\psi_N/D} & \mathbf{G}(L^*/\mathbf{K}^*) \\ \theta\Big\downarrow & & \Big\downarrow \mathrm{res}=\theta' \\ \mathbf{G} & \xrightarrow{\quad\psi\quad} & \mathbf{G}(L/\mathbf{K}) \end{array}$$

It remains to take F the fixed field of $\mathrm{Ker}\,\theta$ in M/E to get a Galois extension F of E such that L is the algebraic closure of K in F and $\mathbf{G}(F/\mathbf{E})$, $\mathrm{res}:\mathbf{G}(F/\mathbf{E})\to\mathbf{G}(L/\mathbf{K})$ are respectively identified with $\mathbf{G}$ and ψ, as contended. □

6.2. Proof of Theorem I. Let $\mathbf{K}=(K;\,P_1,\dots,P_e)$ be a prc e-field. We have to show that $\mathbf{G}(\mathbf{K})$ is projective. According to (4.1) it suffices to show that for every epi

$\psi:\mathbf{G}\to\mathbf{G}(\mathbf{K})$ there exist a coelementary epi $p:\mathbf{T}\to\mathbf{G}(\mathbf{K})$ and a morphism $\theta:\mathbf{T}\to\mathbf{G}$ such that $p=\psi\theta$. Given an epi $\psi:\mathbf{G}\to\mathbf{G}(\mathbf{K})$ it follows by (6.1) that there exist an extension $\mathbf{E}=(E;Q_1,\dots,Q_e)$ of $\mathbf{K}$, with E regular over K and a subfield F of the algebraic closure $\tilde{E}$ of E such that the algebraic closure $\tilde{K}$ of K is contained in F, F is Galois over E, and $\mathbf{G}(F/\mathbf{E})$, $\mathrm{res}:\mathbf{G}(F/\mathbf{E})\to\mathbf{G}(\mathbf{K})$ are respectively identified with $\mathbf{G}$ and ψ. Since, by assumption, $\mathbf{K}$ is a prc e-field, it follows that $\mathbf{K}$ is ec in $\mathbf{E}$ and hence by Scott's lemma [6, Lemma 8.1.3, Corollary 9.3.11], $\mathbf{E}$ can be embedded over $\mathbf{K}$ into an elementary extension $\mathbf{K}^*$ of $\mathbf{K}$. Thus we get the canonical commutative diagram of profinite e-structures

$$\begin{array}{ccc} & & \mathbf{G}(\mathbf{K}^*) \\ & \swarrow^{\theta} & \downarrow p \\ \mathbf{G}=\mathbf{G}(F/\mathbf{E}) & \xrightarrow{\ \psi\ } & \mathbf{G}(\mathbf{K}) \end{array}$$

where the restriction θ is not necessarily an epi. Finally note that the restriction p is a coelementary epi according to (5.1). □

Remark. A tentative to prove the theorem above in the special case $e=1$ is due to McKenna [15] but unfortunately the proof of [15, Theorem 1.1] contains a mistake, though the respective statement is correct. The error occurs at page 1.6, where the Hoschild–Serre sequence contains the incorrect term $H^2(N,u)$ instead of the correct one $H^1(\bar{\pi},H^1(G_K(2),u))$. By contrast with McKenna's intricate approach which requires Galois cohomology, the proof given here is quite simple and of model-theoretic nature.

Finally let us prove the second main result of the paper, which gives a characterization of profinite e-structures which can be realized as absolute Galois e-structures over prc e-fields.

First we prove a little more general result.

6.3. Theorem. *Let $\mathbf{K}=(K;P_1,\dots,P_e)$ be an e-field, L a Galois extension of K such that L is not fr over (K,P_i), $i=1,\dots,e$, $\mathbf{G}$ a profinite e-structure and $\psi:\mathbf{G}\to\mathbf{G}(L/\mathbf{K})$ an epi. Then the next statements are equivalent:*

(i) *There exist an e-field extension $\mathbf{E}$ of $\mathbf{K}$ and an isomorphism $\theta:\mathbf{G}\to\mathbf{G}(\mathbf{E})$ such that $\mathbf{E}$ is a prc e-field, $E\cap L=K$ and the diagram*

$$\begin{array}{ccc} \mathbf{G} & \xrightarrow{\ \theta\ } & \mathbf{G}(\mathbf{E}) \\ & {\scriptstyle\psi}\searrow \quad \swarrow{\scriptstyle\mathrm{res}} & \\ & \mathbf{G}(L/\mathbf{K}) & \end{array}$$

is commutative.

(ii) $\mathbf{G}$ *is projective.*

Proof. (i)→(ii) follows by (6.2).

(ii)→(i). Assume $\mathbf{G}$ is projective. By (6.1), there exist a regular e-field extension $\mathbf{K}'$ of $\mathbf{K}$ and a Galois extension L' of K' such that L is the algebraic closure of K in L' and the restriction res : $\mathbf{G}(L'/\mathbf{K}') \to \mathbf{G}(L/\mathbf{K})$ is identified with the epi ψ. According to [17, Theorem 1.1] there exists a regular e-field extension $\mathbf{M} = (M; Q_1, \ldots, Q_e)$ of $\mathbf{K}'$ such that $\mathbf{M}$ is a prc e-field. Consider the commutative diagram

$$\begin{array}{ccc} \mathbf{G}(\mathbf{M}) & \xrightarrow{\ \lambda\ } & \mathbf{G}(L'/\mathbf{K}') = \mathbf{G} \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & \mathbf{G}(L/\mathbf{K}) & \end{array}$$

where λ and φ are restriction epis. As $\mathbf{G}$ is projective, there exists a mono $\mu : \mathbf{G} \to \mathbf{G}(\mathbf{M})$ splitting λ. Note that for each involution τ of G there is some $x \in \bigcup_{i=1}^{e} X_i(\mathbf{G})$ with $\tau = \sigma(x)$. Indeed $\mu(\tau)$ is an involution of $G(M)$ and hence $\mu(\tau) \in \bigcup_{i=1}^{e} X_i(\mathbf{M})$ since $Q_1, \ldots, Q_e$ are the only orders of M. Assume $\mu(\tau) \in X_i(\mathbf{M})$. Then we get

$$\tau = \lambda^0(\mu(\tau)) = \lambda^0(\sigma(\mu(\tau))) = \sigma(\lambda^i(\mu(\tau)).$$

Thus $\tau = \sigma(x)$ with $x = \lambda^i(\mu(\tau)) \in X_i(\mathbf{G})$.

Let $E \subset \tilde{M}$ be the fixed field of $\mu(G)$. Since $\mu(\mathbf{G})$ is a sub-e-structure of $\mathbf{G}(\mathbf{M})$, we get $\mu(X_i(\mathbf{G})) = \{\sigma_i^\tau \mid \tau \in \mu(G)\}$ for some involution $\sigma_i \in \mu(G)$ for which $\tilde{M}(\sigma_i)$ is a real closure of (M, Q_i), $i = 1, \ldots, e$. Let $Q_i' = E \cap \tilde{M}(\sigma_i)^2$, $i = 1, \ldots, e$. Then $\mu(\mathbf{G})$ is identified with the absolute Galois e-structure of the e-field extension $\mathbf{E} = (E; Q_1', \ldots, Q_e')$ of $\mathbf{M}$. The remark above on the involutions of G implies that there are only e distinct orders on E, namely $Q_1', \ldots, Q_e'$, extending respectively the orders $Q_1, \ldots, Q_e$ of the prc e-field $\mathbf{M}$. Since E is algebraic over the prc field M it follows by [17, Theorem 3.1] that E is prc too, and so $\mathbf{E}$ is a prc e-field. Finally we get the commutative diagram

$$\begin{array}{ccc} \mathbf{G}(\mathbf{E}) = \mu(\mathbf{G}) & \xrightarrow{\ \lambda|_{\mu(\mathbf{G})}\ } & \mathbf{G} \\ & {\scriptstyle\varphi|_{\mu(\mathbf{G})}}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & \mathbf{G}(L/\mathbf{K}) & \end{array}$$

Obviously, $\varphi|_{\mu(\mathbf{G})}$ is epi, i.e. $E \cap L = K$, as contended. □

Remarks. (i) Taking in the statement above $L = \tilde{K}$ and $\psi : \mathbf{G} \to \mathbf{G}(\mathbf{K})$ an epi, it follows that the prc e-field $\mathbf{E}$ from (i) is regular over K.

(ii) It follows from the proof above that for each involution τ of a projective profinite e-structure $\mathbf{G}$ there is $x \in \bigcup_{i=1}^{e} X_i(\mathbf{G})$ with $\tau = \sigma(x)$.

The second main result of the paper is an immediate consequence of (6.3).

6.4. Proof of Theorem II. Let **G** be a profinite *e*-structure. If $\mathbf{G} \cong \mathbf{G}(\mathbf{K})$, **K** a prc *e*-field, then **G** is projective by Theorem I. Conversely, assume **G** is projective, and let **K** be an *e*-field and $L = K(\mathrm{i})$, $\mathrm{i}^2 = -1$. Since **G** is projective, we get by (4.2) an epi $\psi : \mathbf{G} \to \mathbb{Z}_2 \cong \mathbf{G}(L/\mathbf{K})$. Applying (6.3), we get a prc *e*-field **E** with $\mathbf{G}(\mathbf{E}) \cong \mathbf{G}$. □

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